Robust methods for cyclostationarity detection in non-Gaussian signals

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Agenda

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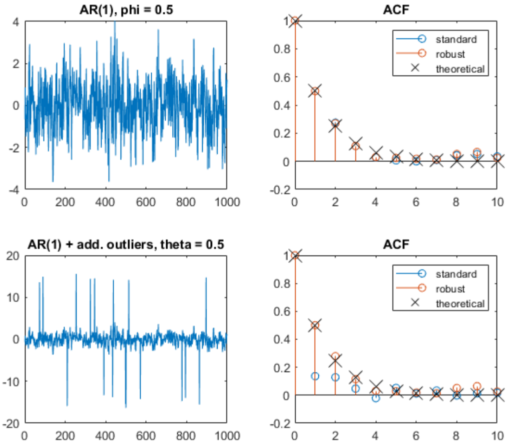
Introduction

- Many local damage detection procedures are based on the cyclostationarity detection methods which utilize the autocorrelation function (ACF) and its standard estimator (sample ACF).
- However, classical methods may fail if the signal of interest is disturbed by heavy-tailed non-Gaussian noise.
- Hence, we propose to use robust ACF estimators in cyclostationarity detection (robust spectral coherence map).
- We also consider robust modifications of coherent and incoherent statistics (i.e. using robust Fourier transform).
- The presented methodologies are applied to simulated and real signals.

Example: ACF in presence of outliers



Example of standard and robust ACF estimation for AR(1) sample without (top) and with additive outliers (bottom).

Robust ACF estimators [2]

Estimation of (Pearson) correlation between centered* vectors $\mathbf{w}_1 = (w_1^1, w_1^2, \cdots, w_1^N)$ and $\mathbf{w}_2 = (w_2^1, w_2^2, \cdots, w_2^N)$:

- (non-robust) sample ACF (numerator: sample ACVF)

$$\boxed{M_1(\mathbf{w}_1, \mathbf{w}_2) = \frac{\frac{1}{N}\sum_{i=1}^N w_1^i w_2^i}{\frac{1}{N}\sqrt{\sum_{i=1}^N (w_1^i)^2 \sum_{i=1}^N (w_2^i)^2}}}$$

- trimmed estimator with parameter $0 \le c < 0.5$ (trimm)

$$\mathbf{w}_3 = (w_3^1, w_3^2, \cdots, w_3^N) = (w_1^1 w_2^1, w_1^2 w_2^2, \cdots, w_1^N w_2^N)$$

$$\tilde{\mathbf{w}}_k = \{w_k^i : i = 1, \ldots, N;\ w_3^{(g)} < w_3^i < w_3^{(n-g+1)}\}, \quad k = 1, 2$$

where $(w_3^{(1)}, \ldots, w_3^{(N)})$ – ordered $\mathbf{w}_3$, $g = \lfloor c \cdot N \rfloor$

$$\boxed{M_2^c(\mathbf{w}_1, \mathbf{w}_2) = M_1(\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_2)}$$

*note: centering is usually done by subtracting the sample mean in non-robust and sample median in robust methods

Robust ACF estimators [2]

- Kendall correlation:

$$\rho_K(\mathbf{w}_1, \mathbf{w}_2) = \frac{2}{N(N-1)} \sum_{1 \le i \le j \le N} \operatorname{sgn}((w_1^i - w_1^j)(w_2^i - w_2^j))$$

$$\boxed{M_3(\mathbf{w}_1, \mathbf{w}_2) = \sin\left(\frac{\pi \rho_K(\mathbf{w}_1, \mathbf{w}_2)}{2}\right)}$$

- Spearman correlation:
  ($\mathbf{r}_1$, $\mathbf{r}_2$ – zero-mean vectors of ranks for $\mathbf{w}_1$, $\mathbf{w}_2$)

$$\rho_S(\mathbf{w}_1, \mathbf{w}_2) = \frac{\sum_{i=1}^N r_1^i r_2^i}{\sqrt{\sum_{i=1}^N (r_1^i)^2 \sum_{i=1}^N (r_2^i)^2}} = M_1(\mathbf{r}_1, \mathbf{r}_2)$$

$$\boxed{M_4(\mathbf{w}_1, \mathbf{w}_2) = \sin\left(\frac{2\pi \rho_S(\mathbf{w}_1, \mathbf{w}_2)}{6}\right)}$$

α-stable distribution and covariation [4]

- Symmetric α-stable distribution $\mathcal{S}(\alpha, \sigma)$ – defined through CF:

$$\Phi_{\mathcal{S}}(z) = \exp\left(-\sigma^{\alpha}|z|^{\alpha}\right), \qquad z \in \mathbb{R},$$

  $\alpha \in (0, 2]$ – stability index, $\sigma > 0$ – scale parameter (infinite variance for $\alpha < 2$, for $\alpha = 2$: Gaussian distribution)

- For two random variables S_1, S_2 from symm. α-stable distribution ($\alpha > 1$), we define the normalized covariation of S_1 on S_2 as

$$NCV(S_1, S_2) = \frac{\mathbb{E}(S_1 \operatorname{sgn}(S_2))}{\mathbb{E}|S_2|}.$$

  Estimation of NCV of centered vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ (sample NCV):

$$\lambda(\mathbf{w}_1, \mathbf{w}_2) = \frac{\sum_{i=1}^{N} w_1^i \operatorname{sgn}(w_2^i)}{\sum_{i=1}^{N} |w_2^i|}$$

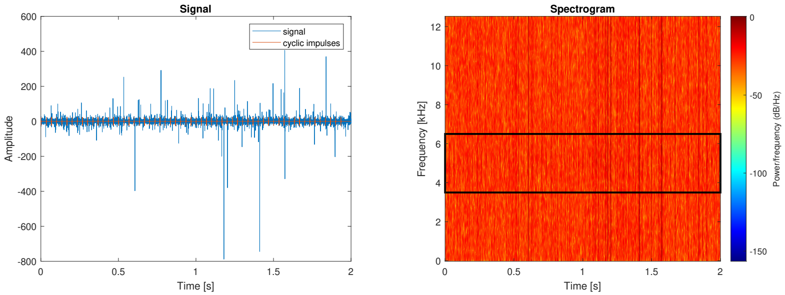
Analyzed simulated and real signals

- Signal 1: signal simulated from the model $\{X_t\}$:

$$X_t = s(t) + Z_t$$

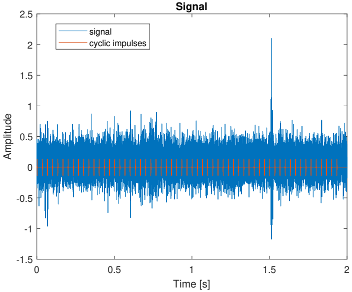
  - $s(t)$ - cyclic impulses
    - cyclic frequency $f_f = 30$ Hz, amplitude $B = 45$
    - informative frequency band $f_c = 3500 - 6500$ Hz
  - $\{Z_t\}$ – sequence of i.i.d. $\mathcal{S}(\alpha = 1.7, \sigma = 3)$ random variables
- Signal 2: real vibration signal from a (healthy) crushing machine with added cyclic impulses $s(t)$ (of amplitude $B = 0.25$)
- both signals consist of $L = 50000$ observations (sampling frequency 25000 Hz, 2 seconds)

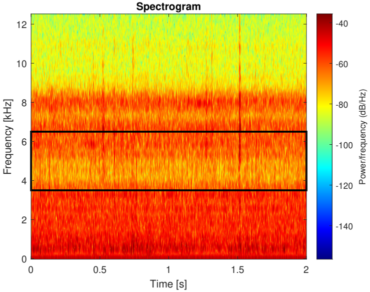
Analyzed simulated and real signals: Signal 1



Signal 1 and its spectrogram.

Analyzed simulated and real signals: Signal 2





Signal 2 and its spectrogram.

Robust spectral coherence maps

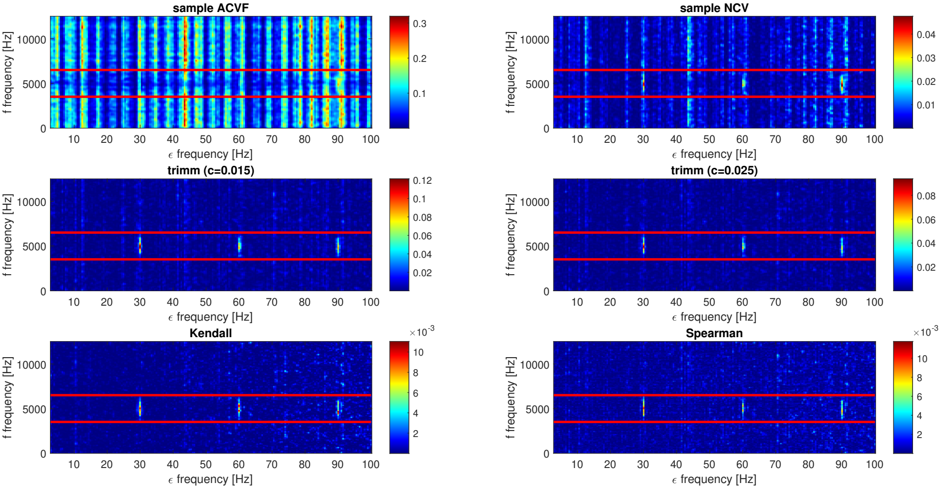
Frequency-frequency domain analysis – spectral coherence map $|\gamma(f,\epsilon)|^2$ (ϵ – cycle frequency) [1, 3, 4]

Algorithm 2 Robust spectral coherence for a signal $X = x_1, \dots, x_L$.

- $\odot$ - element-wise multiplication of vectors
- $X[\text{index}] = [X_{\text{index}_1}, \dots, X_{\text{index}_n}]$, where $\text{index} = [\text{index}_1, \dots, \text{index}_n]$

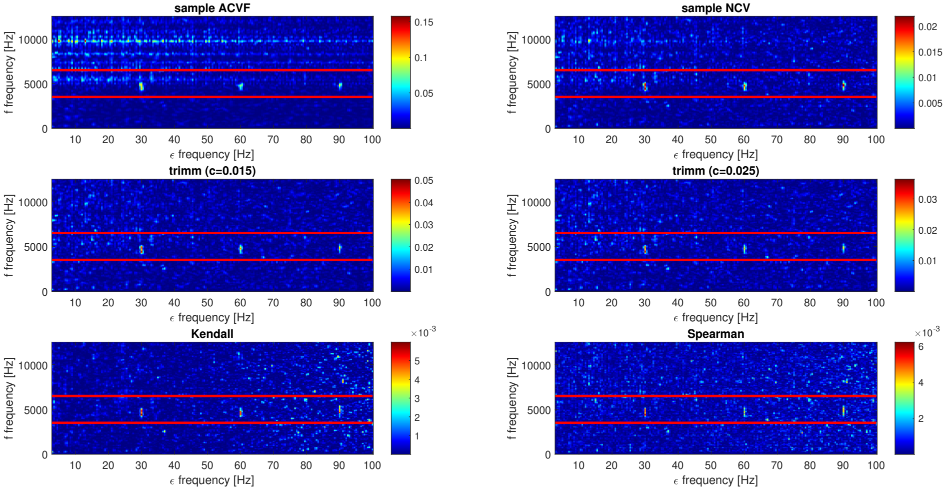
Set $M(\cdot,\cdot)$ - selected robust covariance/correlation estimator
Set $w(\cdot)$ - window function of length n
Set nfft - number of sampling points to calculate DFT
Set nover - size of overlap
Set ϵ_{min}, ϵ_{max} – minimal and maximal modulation frequency
$t = [0, 1, \dots, N-1]$
for $k \leftarrow \epsilon_{\text{min}}$ to ϵ_{max} **do**
    $K = \left\lfloor \frac{N - \text{nover}}{n - \text{nover}} \right\rfloor$
    $X^k = X \odot e^{i\pi k t}$
    $Y^k = X \odot e^{-i\pi k t}$
    $\text{index} = [1, \dots, n]$
    for $i \leftarrow 1$ to K **do**
        $X^w = w \odot X^k[\text{index}]$
        $Y^w = w \odot Y^k[\text{index}]$
        for $j \leftarrow 1$ to nfft **do**
            $X_w(j,i) = \text{DTFT}_{\text{nfft}}(j, X^w)$
            $Y_w(j,i) = \text{DTFT}_{\text{nfft}}(j, Y^w)$
        end for
        $\text{index} = \text{index} + (n - \text{nover})$
    end for
    for $j \leftarrow 1$ to nfft **do**
        $S_X(f_j, \epsilon_k) = M(Y_w(j,:), X_w(j,:)^*)$
    end for
    Calculate robust spectral coherence $|\gamma_X(f_j, \epsilon_k)|^2$
end for

Robust spectral coherence maps



Spectral coherence maps $|\gamma(f, \epsilon)|^2$ for Signal 1.

Robust spectral coherence maps



Spectral coherence maps $|\gamma(f,\epsilon)|^2$ for Signal 2.

Robust spectral coherence maps

To compare the values of periodic impulses (in f_c band) with the noise on the map, we calculate amplitude ratio for each ϵ

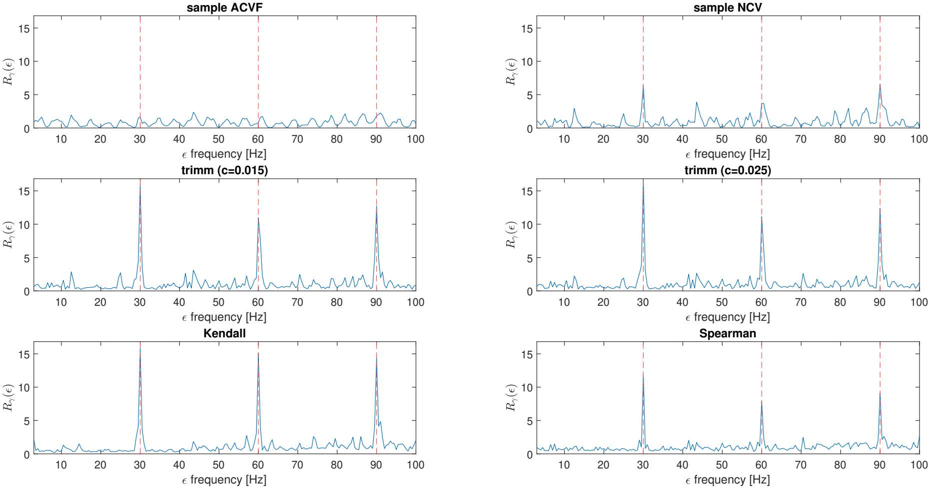
$$R_\gamma(\epsilon) = \frac{\overline{|\gamma(f_c, \epsilon)|^2}}{\overline{|\gamma|^2}},$$

where $\overline{|\gamma(f_c, \epsilon)|^2}$ is the mean of map values in f_c band for ϵ, and $\overline{|\gamma|^2}$ is the mean of all map values.

For an evaluation of the map, we consider the following indicator:

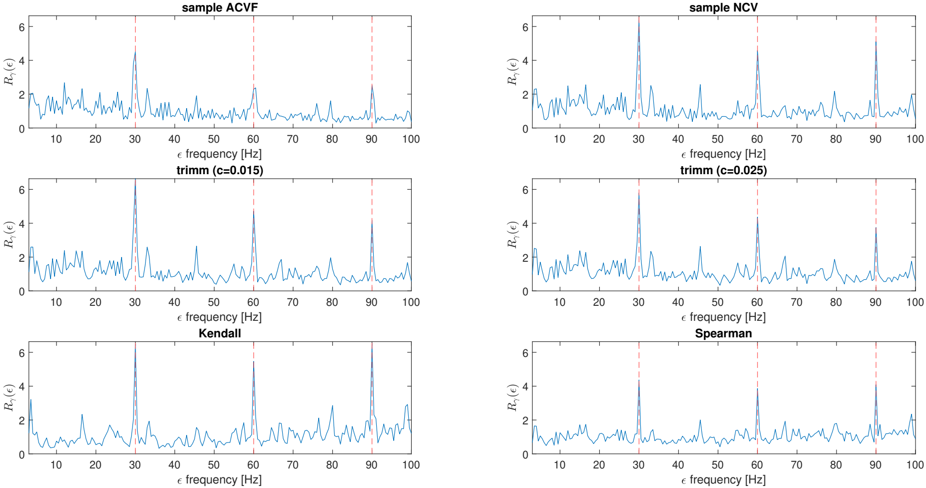
$$\tau_\gamma = \frac{\sum_{\epsilon \text{ cyclic}} R_\gamma(\epsilon)}{\sum_{\epsilon=\epsilon_{min}}^{\epsilon_{max}} R_\gamma(\epsilon)}$$

Robust spectral coherence maps



Amplitude ratios $R_\gamma(\epsilon)$ for Signal 1.

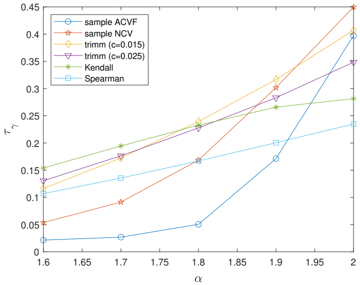
Robust spectral coherence maps



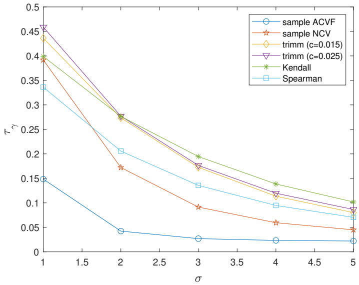
Amplitude ratios $R_\gamma(\epsilon)$ for Signal 2.

Robust spectral coherence maps

$\{Z_t\} \sim \mathcal{S}(\alpha, \sigma = 3)$

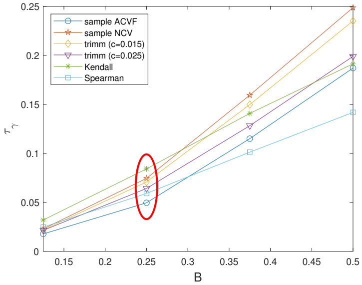


$\{Z_t\} \sim \mathcal{S}(\alpha = 1.7, \sigma)$



Values of τ_γ calculated for Signal 1 with different $\{Z_t\} \sim \mathcal{S}(\alpha, \sigma)$ cases.

Robust spectrogram-based autocorrelation maps



Values of τ_γ calculated for Signal 2 with different amplitudes B of added cyclic impulses.

Robust coherent/incoherent statistics for spectrograms

Sample coherence [5]:

$$|\hat{\gamma}(p,q,M)|^2 = \frac{|\sum_{m=0}^{M-1} I(\omega_{p+m})\overline{I(\omega_{q+m})}|^2}{\sum_{m=0}^{M-1}|I(\omega_{p+m})|^2 \sum_{m=0}^{M-1}|I(\omega_{q+m})|^2} \quad (I(\omega_j) \text{ is DFT})$$

Coherent statistic:

$$|\hat{\gamma}(0,d,N)|^2$$

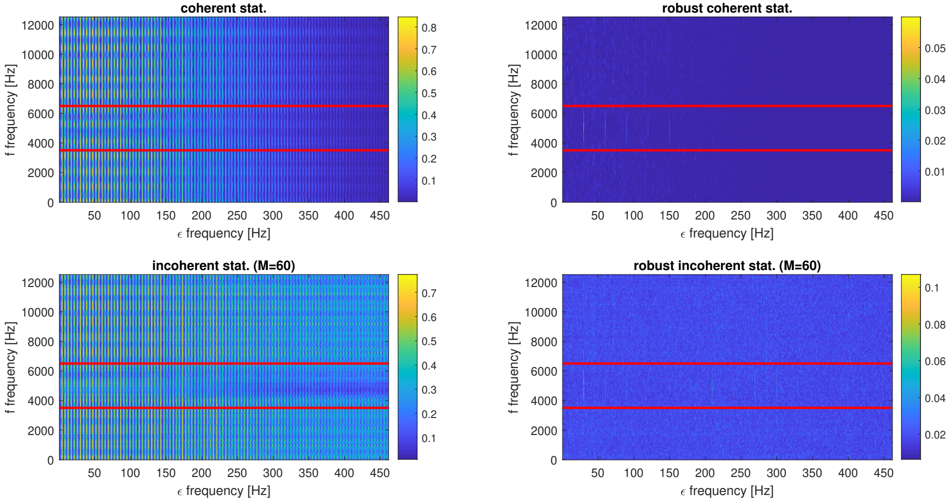
Incoherent statistic:

$$\delta(d,M) = \frac{1}{L+1}\sum_{p=0}^{L}|\hat{\gamma}(pM, pM+d, M)|^2 \quad (L = [(N-1-d)/M])$$

Idea:

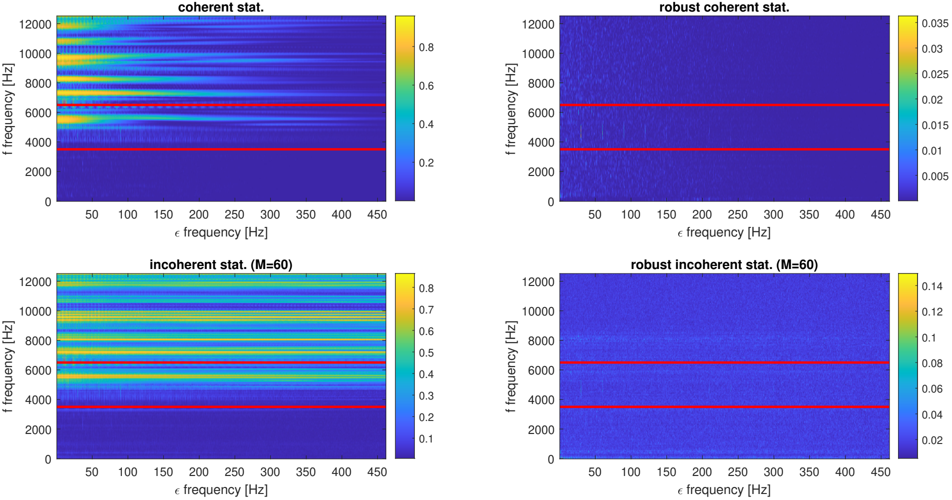
- replace Fourier transform with robust Fourier transform (using M-regression [6])
- apply coh./incoh. statistics to time series from spectrogram

Robust coherent/incoherent statistics for spectrograms



Robust coherent/incoherent statistic maps for Signal 1.

Robust coherent/incoherent statistics for spectrograms



Robust coherent/incoherent statistic maps for Signal 2.

Summary

- In this work, we presented the application of robust statistics for selected cyclostationarity detection methods (spectral coherence maps, coherent/incoherent statistics).
- As presented, the proposed approach may outperform classical (non-robust) methods when the non-Gaussian behaviour of the analyzed signal is present.
- In practice, such behaviour may occur due to specific processes conducted by the machine (e.g. cutting, crushing, drilling).
- We plan to further develop robust methods for local damage detection and to apply them to other real datasets.

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Thank you for your attention!